Vector Modes of Cosmological Perturbations

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Considered metrices $g_{\mu\nu}$ and $f_{\mu\nu}$ for bimassive gravity:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \varepsilon g^{(1)}_{\mu\nu}$$
$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = a^2 \left(-dt^2 + \delta_{ij} dx^i dx^j\right)$$
$$g^{(1)}_{\mu\nu} dx^\mu dx^\nu = a^2 \left(-2A dt^2 - 2B_i dx^i dt + 2H_{ij} dx^i dx^i\right)$$

$$f_{\mu\nu} = f^{(0)}_{\mu\nu} + \varepsilon f^{(1)}_{\mu\nu}$$
$$f^{(0)}_{\mu\nu} dx^\mu dx^\nu = b^2 \left(-\frac{\mathcal{H}^2}{\mathcal{H}_f^2} dt^2 + \delta_{ij} dx^i dx^j\right)$$
$$f^{(1)}_{\mu\nu} dx^\mu dx^\nu = b^2 \left(-2\mathscr{A}\frac{\mathcal{H}_f^2}{\mathcal{H}^2} dt^2 - 2\mathscr{B}_i \frac{\mathcal{H}_f}{\mathcal{H}} dx^i dt + 2\mathscr{H}_{ij} dx^i dx^i\right)$$

where $\mathcal{H} \equiv \frac{\dot{a}}{a}$ and $\mathcal{H}_f \equiv \frac{\dot{b}}{b}$.
At the beginning, I thought that it might be usefull for next calculations to define $\mathscr{A}$, $\mathscr{B}_i$ and $\mathscr{H}_{ij}$, describing cosmological perturbations with the ornaments in the form: $\left(\frac{\mathcal{H}}{\mathcal{H}_f}\right)^2$, resp. $\frac{\mathcal{H}}{\mathcal{H}_f}$.

Keeping on mind the form of the considered metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ we can find the components of $X^{\mu}{}_{\nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$ in the linear order of perturbation theory:

$$\mathbb{X} = \mathbb{X}^{(0)} + \varepsilon \mathbb{X}^{(1)}$$

$$\mathbb{X}^{(0)} = diag\left(r\frac{\mathcal{H}}{\mathcal{H}_f}, r, r, r \right)$$

where $r \equiv \frac{b}{a}$.

The components of the $\mathbb{X}^{(1)}$:

$$X^{0}{}_{0} = \frac{\mathcal{H}_f}{\mathcal{H}} \frac{(\mathcal{A} - A)}{r} \qquad X^{0}{}_{i} = r\frac{\frac{\mathcal{H}_f}{\mathcal{H}}\mathcal{B}_i - B_i}{\frac{\mathcal{H}_f}{\mathcal{H}} + 1}$$

$$X^{i}{}_{0} = r\frac{\frac{\mathcal{H}_f}{\mathcal{H}} B_i - \mathcal{B}_i}{\frac{\mathcal{H}}{\mathcal{H}_f} + 1} \qquad X^{i}{}_{j} = r\left(\mathcal{H}_{ij} - H_{ij}\right)$$

Under the gauge transformations:

$$g^{(1)} \to g^{(1)} + a^{-2}\mathcal{L}_X g^{(0)} \qquad f^{(1)} \to f^{(1)} + a^{-2}\mathcal{L}_X f^{(0)}$$

considering the transformation of vector perturbations[1]:

$$B^{(V)} \to B^{(V)} - \dot{L}^{(V)} \qquad \frac{\mathcal{H}_f}{\mathcal{H}}\mathscr{B}^{(V)} \to \frac{\mathcal{H}_f}{\mathcal{H}}\mathscr{B}^{(V)} - \dot{L}^{(V)}$$

$$H^{(V)} \to H^{(V)} - kL^{(V)} \qquad \mathscr{H}^{(V)} \to \mathscr{H}^{(V)} - kL^{(V)}$$

Constructing gauge invariant variables:

$$\Delta B^{(V)} = B^{(V)} - \frac{\mathcal{H}_f}{\mathcal{H}}\mathscr{B}^{(V)} \qquad \Delta H^{(V)} = H^{(V)} - \mathscr{H}^{(V)}$$

$$\sigma^{(V)} = k^{-1}\dot{H}^{(V)} - B^{(V)} \qquad \chi^{(V)} = k^{-1}\dot{\mathscr{H}}^{(V)} - \frac{\mathcal{H}_f}{\mathcal{H}}\mathscr{B}^{(V)}$$

[1] After decomposing the metric pertubations into the scalar, vector and tensor modes.

Modified Einstein equations for bimassive gravity in the first order of perturbation theory describing vector perturbations lead to two constraints between gauge invariant variables:

$$k^2\sigma^{(V)} = 2a^2\left(r\mathcal{B}_2\frac{\Delta B^{(V)}}{1+\frac{\mathcal{H}_f}{\mathcal{H}}} - \Omega\left(\overline{\rho}+\overline{P}\right)\right)$$

$$k^2\chi^{(V)}\frac{\mathcal{H}_f}{\mathcal{H}} = -2a^2\left(\frac{\mathcal{B}_2}{r}\frac{\Delta B^{(V)}}{1+\frac{\mathcal{H}_f}{\mathcal{H}}}\right)$$

and two dynamic equations:

$$\dot{\sigma}^{(V)} + 2\mathcal{H}\sigma^{(V)} = \frac{a^2}{k}\left(\Pi^{(V)}\overline{P} - r\left(\mathcal{B}_4 + r\frac{\mathcal{H}_f}{\mathcal{H}}\mathcal{B}_5\right)\Delta H^{(V)}\right)$$

$$\dot{\chi}^{(V)} + 2\mathcal{H}_f\chi^{(V)} = \frac{a^2}{kr}\left(2\mathcal{B}_2 + \mathcal{B}_4 + r\frac{\mathcal{H}_f}{\mathcal{H}}\left(3\mathcal{B}_5 + 2\mathcal{B}_6 r\right)\right)\Delta H^{(V)}$$

where:

$$\mathcal{B}_2 = \beta_1 + 2\beta_2 r + \beta_3 r^2 \qquad \mathcal{B}_4 = \beta_1 + r\beta_2$$

$$\mathcal{B}_5 = \beta_2 + r\beta_3 \qquad \mathcal{B}_6 = \beta_3 + r\beta_4.$$

Using the modified Einstein equations in the $0-th$ order, we can express ratio of Hubble rates in the form:

$$\frac{\mathcal{H}_f}{\mathcal{H}} = \frac{\overline{\rho} + \mathcal{B}_0 + 3\left(\overline{P} + \frac{\mathcal{B}_2}{r} - \mathcal{B}_3 - r\mathcal{B}_4\right)}{\overline{\rho} + \mathcal{B}_0 + 3\left(r\mathcal{B}_2 - \mathcal{B}_5 - r\mathcal{B}_6\right)},$$

where aditionally:

$$\mathcal{B}_0 = \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \qquad \mathcal{B}_3 = \beta_0 + r\beta_1$$

Equations in the 0 th-order of perturbation theory:

$$\mathcal{H}^2 = \frac{a^2}{3}\left(\overline{\rho} + \mathcal{B}_0\right), \quad \mathcal{H}^2 - 2\frac{\ddot{a}}{a} = a^2\left(\overline{P} - \mathcal{B}_3 - \mathcal{B}_4 - \frac{\mathcal{H}_f}{\mathcal{H}} r\mathcal{B}_2\right) \tag{1}$$

$$\mathcal{H}^2 = \frac{a^2}{3r}\mathcal{B}_1, \qquad \mathcal{H}^2 - \frac{\ddot{a}}{a} = -\frac{a^2}{2}\left(\frac{\mathcal{B}_2}{r} + \frac{\mathcal{H}_f}{\mathcal{H}}\left(\mathcal{B}_5 + r\mathcal{B}_6 - \frac{\mathcal{B}_1}{3r}\right)\right) \tag{2}$$

Next, we concentrate on the solution for the viable cosmological bimassive gravity theory ($\beta_2 = \beta_3 = 0$). In such a case: $\mathcal{B}_2 = \beta_1$, $\mathcal{B}_4 = \beta_1$, $\mathcal{B}_5 = 0$, $\mathcal{B}_6 = \beta_4 r$. Next, we can easily eliminate $\Delta B^{(V)}$ from the first two constraining equations, obtaining:

$$\chi^{(V)} = -\frac{1}{r^2}\frac{\mathcal{H}}{\mathcal{H}_f}\left(\sigma^{(V)} + \frac{2a^2}{k^2}\Omega\left(\overline{\rho} + \overline{P}\right)\right) \tag{3}$$

We can also eliminate $\delta H^{(V)}$ from the second two dynamic equations:

$$\left(\dot{\sigma}^{(V)} + 2\mathcal{H}\sigma^{(V)} - \frac{a^2}{k}\Pi^{(V)}\overline{P}\right) \times \\ \left(3\frac{\beta_1}{r} + \frac{\mathcal{H}_f}{\mathcal{H}}2r^2\beta_4\right) + \left(\dot{\chi}^{(V)} + 2\mathcal{H}_f\chi^{(V)}\right) r\beta_1 = 0 \tag{4}$$

For our following idea, it shows to be appropriate to combine both (00) parts of modified Einstein equations. In order of ε^0, we are left with:

$$\beta_4 r^3 - 3\beta_1 r^2 - (\overline{\rho} + \beta_0)\, r + \beta_1 = 0 \tag{5}$$

We also recall the expresion for the $\mathcal{H}_f / \mathcal{H}$ which obeys a form:

$$\frac{\mathcal{H}_f}{\mathcal{H}} = \frac{3\beta_1 \left(1 - r^2\right) + \left(\overline{\rho} - 2\beta_0 + 3\overline{P}\right) r}{-3\beta_4 r^3 + 6\beta_1 r^2 + \left(\beta_0 + \overline{\rho}\right) r} \tag{6}$$

Next, in purpose to study effects from the considered interactions between metrices g and f hidden in the introduced tensor $B_{\mu\nu}$ let us consider unrealistic toy model with: $\overline{\rho} = \overline{P} = 0$. Using this assumption and considering (8) as well as (8) we can easily see that $r(\eta) = const.$, which means $\dot{r} = 0$. Next, we also realize that:

$$\frac{\mathcal{H}_f}{\mathcal{H}}(t) = \frac{\mathcal{H}_f}{\mathcal{H}} \left(r\left(t\right)\right), \tag{7}$$

and so $\dot{\left(\mathcal{H}_f / \mathcal{H}\right)} = 0$.

Our constrained equation (7), together with its time derivative are:

$$\chi^{(V)} = -\frac{1}{r^2}\frac{\mathcal{H}}{\mathcal{H}_f}\sigma^{(V)} \qquad \dot{\chi}^{(V)} = -\frac{1}{r^2}\frac{\mathcal{H}}{\mathcal{H}_f}\dot{\sigma}^{(V)} \tag{8}$$

Next, we are going to concentrate only on cases without cosmological constant $\beta_0 = 0$. It means that only two nonzero β parameters are β_1 and β_4. Using (8) and (7) we can imediatelly see that under this assumption:

$$\mathcal{H}_f = \mathcal{H} \tag{9}$$

Combining (9), (8), (9) together with the assumption $P = 0$ and the dynamic equation (7). we are left with:

$$\dot{\sigma}^{(V)} + 2\mathcal{H}\sigma^{(V)} = 0 \tag{10}$$

$$\chi^{(V)} = -\frac{1}{r^2}\sigma^{(V)} \tag{11}$$

Since r is just a constant, these equations have trivial solution:

$$\sigma^{(V)} = \frac{\sigma_I^{(V)}}{a^2} \qquad\qquad \chi^{(V)} = \frac{\sigma_I^{(V)}}{b^2} \tag{12}$$

There are at least two things, which we should mention together with the presented solution. At first, In Unverse with no matter or rad. component, the vector modes of the cosmological perturbations for bimassive gravity theory will be decreasing function of a or b, respectively.

Doteraz dosiahnuté výsledky pri riešení prezentovaného problému boli dosiahnuté s podporou Ministerstva školstva, vedy výskumu a športu SR v rámci poskytnutia dotácie v zmysle § 8a zákona č.172/2005 Z. z. o organizácii štátnej podpory výskumu a vývoja a o doplnení zákona č. 575/2001 Z. z. o organizácii činnosti vlády a organizácii ústrednej štátnej správy v znení neskorších predpisov v platnom znení.